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The global long-range order of quasi-periodic patterns in Islamic architecture

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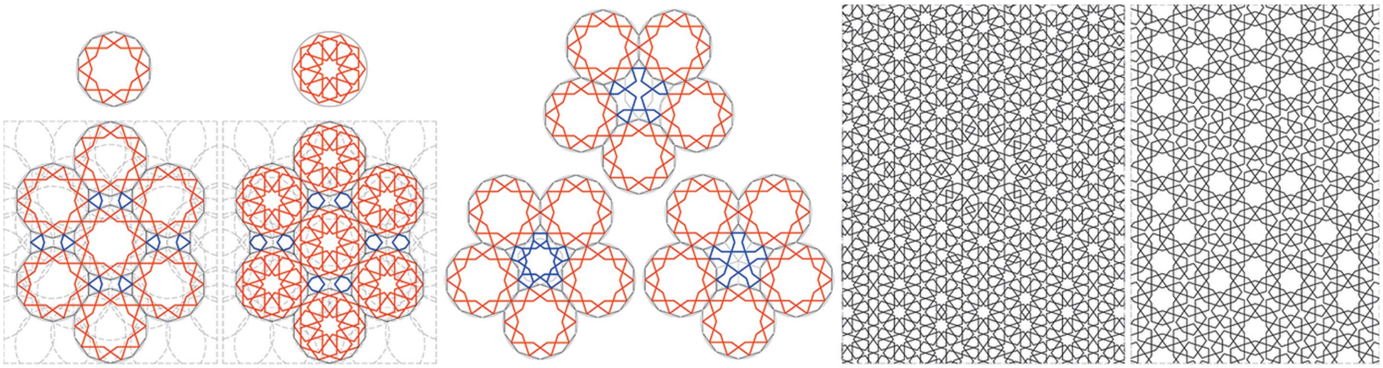
Three decades after their discovery, the unique long-range structure of quasicrystals still poses a perplexing puzzle. The fact that some ancient Islamic patterns share similar quasi-periodic symmetries has prompted several scientists to investigate their underlying geometry and construction methods. However, available structural models depend heavily on local rules and hence they were unable to explain the global long-range order of Islamic quasi-periodic patterns. This paper shows that ancient designers, using simple consecutive geometry, have resolved the complicated long-range principles of quasi-periodic formations. Derived from these principles, a global multi-level structural model is presented that is able to describe the global long-range translational and orientational order of quasi-periodic formations. The proposed model suggests that the position of building units, locally and globally, is defined by one framework, and not tiled based on local rules (matching, overlapping or subdividing). In this way, quasi-periodic formations can grow rapidly *ad infinitum* without the need for any defects or mismatches. The proposed model, which presents a novel approach to the study of quasi-periodic symmetries, will hopefully provide a deeper understanding of the structure of quasicrystals at an atomic scale, allowing scientists to achieve improved control over their composition and structure.

1. Introduction

The unexpected discovery of quasicrystals in the early 1980s attracted significant scientific interest because of their unusual structural properties, exhibiting symmetries long thought forbidden in classical crystallography (Shechtman *et al.*, 1984). The atoms in these unusual structures are neither arranged in neat rows at regularly spaced intervals, similar to crystals, nor scattered randomly, similar to glass. Instead, they exhibit a complicated long-range translational order that is not periodic and a long-range orientational order (Levine & Steinhardt, 1986; Yamamoto & Takakura, 2008). To understand the unusual structural properties of quasicrystals, scientists turned to alternative structural models. One early model for describing quasicrystals was based on a tiling discovered by mathematical physicist Roger Penrose in the 1970s (Penrose, 1974). These tiling patterns consist of two differently shaped tiles that join neatly, according to local matching rules, to cover a flat surface completely. Quasi-periodic patterns can also be generated mathematically using the inflation–deflation operation (De Bruijn, 1981*a*), the grid method (De Bruijn, 1981*b*), the strip projection method (Kramer, 1982), the cut projection method (Bak, 1986) and the generalized dual method (Socolar *et al.*, 1985). Although these structural models provide important insights into understanding the structure of quasi-periodic patterns, there is still significant information lacking concerning the determination of their long-range order.

The discovery of ancient Islamic patterns with quasi-crystalline structural properties has triggered significant discussion and a number of debates on the scientific relevance of Islamic geometry. To date, three types of Islamic quasi-periodic patterns were documented in Islamic historical ornaments. These include octagonal (Makovicky & Fenoll Hach-Alí, 1996), decagonal (Makovicky, 1992, 2007, 2008; Makovicky *et al.*, 1998; Rigby, 2005; Lu & Steinhardt, 2007*a*; Saltzman, 2008) and dodecagonal (Makovicky & Makovicky, 2011). The striking similarities between these quasi-periodic Islamic patterns and the unique puzzling order of quasicrystals have triggered significant research into understanding the structural principles of Islamic formations. However, none of these investigations were able to describe the global long-range principles of quasi-periodic patterns in Islamic architecture.

This paper presents the first global structural model that is able to describe the long-range translational and orientational order of quasi-periodic formations in Islamic architecture. The method is used to construct infinite formations of perfect quasicrystalline patterns, including perfect Penrose tilings, without using confusing strategies or complicated mathematics.



(a) (b) (c) (d) (e)

Figure 1
The types of Islamic patterns. (*a*), (*b*) Two different periodic patterns constructed by generating different star units within the same underlying decagon basic grid. (*c*) Three different line variations for connecting the same star units. (*d*), (*e*) Two examples of quasi-periodic patterns found in Islamic architecture. (*d*) Quasi-periodic pattern found in the courtyard of the Madrasa of al-'Attarin (1323) in Fez, Morocco. (*e*) Quasi-periodic pattern found on the walls of the Darb-i Imam shrine (1453) in Isfahan, Iran.

2. Background

Geometry is one of the chief characteristics that give the Islamic artistic tradition its distinct identity. This tradition was completely inspired by a deep religious, philosophical and cosmological approach, which embodied all aspects of life and manifested itself in every product (Kritchlow, 1976; Al-Bayati, 1981). The vast variety of geometric formations and the strict rules of its generation reveal an important inner dimension of Islamic tradition: '*unity in multiplicity and multiplicity in unity*' (Kritchlow, 1976; Al-Bayati, 1981; Jones, 1978). Islamic artists did not seek to express themselves, but rather aimed to honor matter and reveal the objective nature of its meaning (Kritchlow, 1976; Jairazbhoy, 2000). This principle is represented by means of using the same proportional systems that nature embodies, which underlie the geometry of architectural spaces and geometric patterns (Kritchlow, 1976; Ritchard, 2007).

Traditionally, Islamic patterns were constructed by using a compass and a straightedge; therefore the generating force of patterns lies in the center of the circle (Kritchlow, 1976; Jones, 1978; El-Said, 1993). These complex geometric formations are elaborations of simple constructions of circles, which are often used to determine the underlying basic grids. Mathematically these grids are known as tessellations, in which polygons are repeated to fill the plane (Gonzalez, 2001). The vast varieties of ornamental compositions are achieved by developing different star units within different variations of the basic grids (Kritchlow, 1976; El-Said, 1993; Al Ajlouni, 2009). These star units are created by proportionally breaking down these polygons to form the different designs. The final formations are then developed through intelligent extension of parallels forming a network of lines connecting the main unit. These connecting formations can take different designs without affecting the symmetry of the overall pattern. Figs. 1(*a*) and 1(*b*) show two different patterns constructed by generating different star units within the same underlying decagon basic grid. Fig. 1(*c*) shows three different line variations for connecting the same star units.[1]

Interestingly, some Islamic patterns exhibit a more chaotic arrangement of elements with the presence of some local order but unclear long-range order (Makovicky, 1992, 2007, 2008; Makovicky *et al.*, 1998; Lu & Steinhardt, 2007*a*; Rigby, 2005; Makovicky & Makovicky, 2011; Saltzman, 2008). Examples of these patterns can be found on the Madrasa of al-'Attarin (1323) in Fez, Morocco (Fig. 1*d*) and Darb-i Imam shrine (1453) in Isfahan, Iran (Fig. 1*e*). The aperiodic order of these patterns is somewhat similar to the structural signature of quasicrystals. In their attempts to resolve the structural order of Islamic quasi-periodic patterns, scientists investigated the relationship between these patterns and Penrose tiling patterns (Makovicky, 1992, 2008; Makovicky *et al.*, 1998; Lu & Steinhardt, 2007*a*; Zaslavsky *et al.*, 1991; Chorbachi, 1989; Bonner, 2003). Makovicky developed new variations of the Penrose tiles based on his analyses of the patterns on the external walls of the Gunbad-I Kabud tomb tower in Maragha, Iran (1197) (Makovicky, 1992) and later, with colleagues, investigated the relations between Penrose-type tiling and traditional Islamic ornaments in Spain and Morocco (Makovicky *et al.*, 1998). Bonner has investigated three styles of self-similarity in 14th and 15th century Islamic geometric ornament (Bonner, 2003). In 2007, Lu & Steinhardt (2007*a*,*b*,*c*), based on their investigation of the patterns on the Darb-i Imam shrine (1453) in Isfahan, Iran, suggested that Islamic decagonal quasi-periodic patterns were constructed by combining a localized tiling approach of a special set of decorated '*girih tiles*' with self-similar transformations. Most recently, Makovicky & Makovicky (2011) suggested that the dodecagonal quasi-periodic star pattern found in Fez is based on the Amman quasi-lattice grid.

Although the body of literature provides important insights into understanding the local properties of Islamic quasi-periodic patterns, nevertheless, all proposed methods so far are based on localized rules (*e.g.* subdividing, matching or overlapping) and were unable to describe the long-range

[1] Larger versions of all of the figures presented in this paper have been deposited in the IUCr electronic archives (Reference: DM5019). Services for accessing these data are described at the back of the journal.

order of quasi-periodic patterns in Islamic architecture. In addition, none of these methods could be generalized beyond their specific case studies. As a result, many scientists have concluded that the Muslim designers constructed these patterns with localized tiling systems and without being aware of their global long-range order (Makovicky, 2007; Lu & Steinhardt, 2007c; Cromwell, 2009; Bohannon, 2007). Unfortunately, the suggested tiling approach, which completely disagrees with the generating principles and philosophy of Islamic geometry (Kritchlow, 1976), had doubted the architectural ability of the ancient Muslim designers and, accordingly, evaluated both the processes and the outputs of their creativity. The fact that Muslim designers were able to construct a wide variety of quasi-periodic patterns suggests that they have used a clear and consistent formal method to design and implement these complicated formations.

3. Solving the puzzle

Derived from the traditional principles of Islamic geometry and based on my examination of a large number of Islamic patterns, I present the first global multi-level hierarchical framework model (HFM) that is able to describe the long-range translational and orientational order of quasi-periodic formations in Islamic architecture. I further demonstrate how this model can be used to construct and grow infinite formations of quasi-periodic patterns and more importantly demonstrate the universal applicability of this method to generate perfect quasicrystalline formations, including infinite perfect Penrose tilings.

The proposed model conforms to the traditional Islamic method of using a compass and straightedge, in which the generating force of patterns lies in the center of the circle. It is derived from the principle that Islamic patterns are based on a combination of an underlying basic grid and a star unit (Kritchlow, 1976; El-Said, 1993; Al Ajlouni, 2009). While the basic grid is used to define the type of symmetry by defining the positions of star units within the overall formation, the star unit defines the internal variations of the patterns' design, without affecting the overall symmetry. Accordingly, the underlying basic grid is the key to resolving the order of Islamic quasi-periodic patterns.

As an illustration of this model, consider the quasi-periodic cartwheel pattern in Fig. 2(*f*), which was commonly used in Seljuk architecture (Schneider, 1980) [*e.g.* the Darb-i Imam shrine (1453) and the Friday Mosque in Isfahan, Iran]. The full sequence of constructing the quasi-periodic cartwheel pattern is demonstrated in Fig. 2. A framework of nested decagrams (Fig. 2*a*) serves as the underlying basic grid. The framework grows based on the Fibonacci sequence. As we add more and more decagrams, the ratio between the radii of any two successive decagrams is equal to the golden ratio. If we denote the radius of the nth decagram by r_n and the next larger radius by r_{n+1}, then the ratio r_{n+1}/r_n is equal to the golden ratio $\varphi = (1 + 5^{1/2})/2$. All

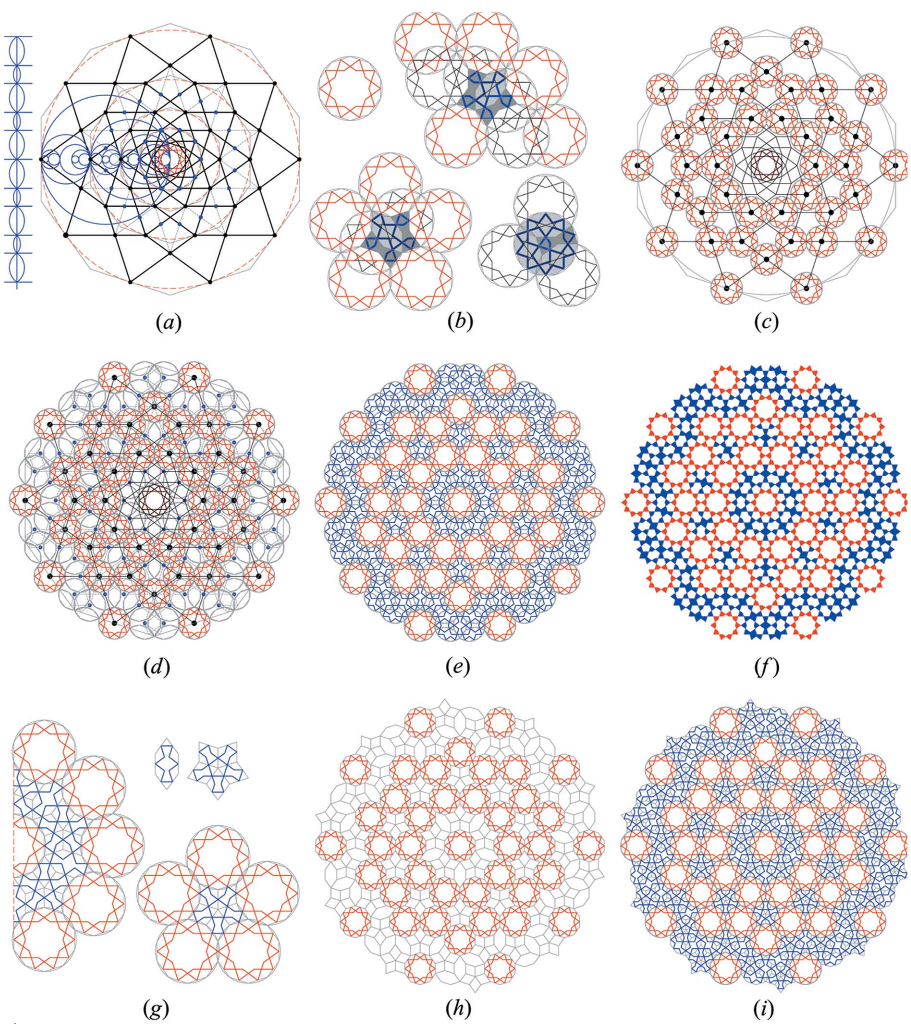


Figure 2
The sequence of constructing the cartwheel pattern of the first-level hierarchy of the global quasi-periodic empire. (*a*) A framework of the nested decagrams, which grows based on the Fibonacci sequence, serves as the underlying basic grid for the quasi-periodic pattern. The size of the central star 'seed' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams. (*b*) The main star unit and the connecting formations. (*c*) The positions of star units are determined entirely by the network of the nested decagrams. The black dots correspond to the center position of all instances of the star unit. (*d*) The connecting formations are formed by overlapping the main units. (*e*) The final line configuration of the first-level cartwheel of the global quasi-periodic empire. (*f*) The final rendered pattern of the first-level hierarchy of the global quasi-periodic empire. (*g*) A different possible connecting arrangement, which is constructed by extending the lines of the main star units to meet. (*h*), (*i*) The process of constructing a new variation of the cartwheel pattern.

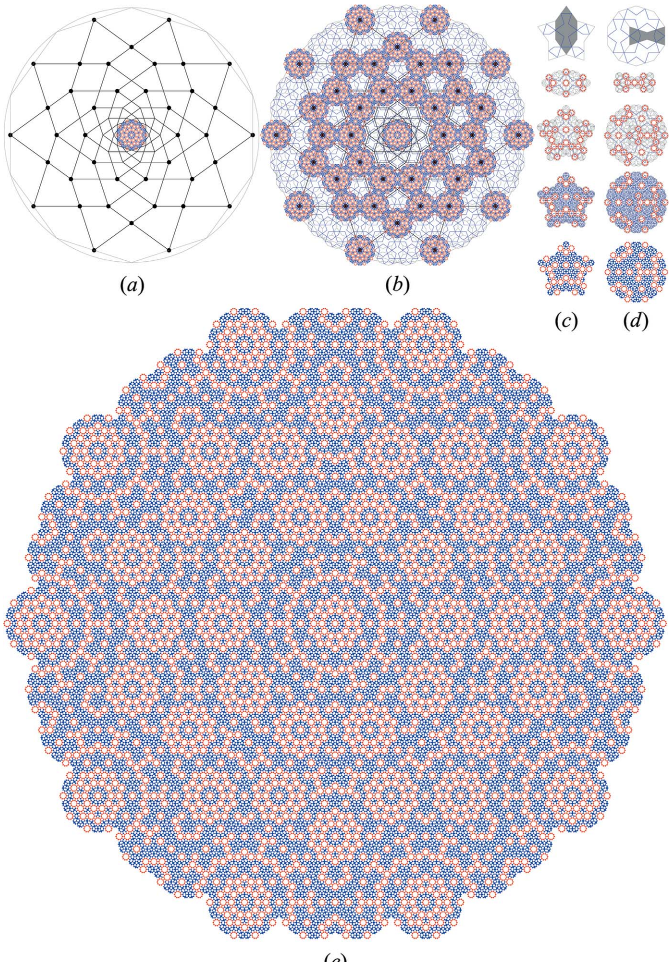


Figure 3
The sequence of constructing the pattern of the second-level hierarchy of the global quasi-periodic empire. (*a*) A new generation of the framework of the nested decagrams serves as the underlying basic grid for the second-level quasi-periodic pattern. The central star 'seed' unit is the same final cartwheel pattern in Fig. 2(*f*). (*b*) The distribution of the main cartwheel units and their connecting formations are determined entirely by the network of the nested decagrams. (*c*), (*d*) The two connecting formations used to fill the gaps between the main star units. (*e*) The final rendered pattern of the second-level hierarchy of the global quasi-periodic empire.

dimensions within this sequence are related to each other by the golden ratio proportional system.

According to the HFM model, the quasi-periodic empire is generated around one main 'center of origin', the center of the global tenfold proportional system. The size of the central star unit '*seed*' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams (Fig. 2*a*). The positions of star units are determined entirely by the network of the nested decagrams. The black dots in Fig. 2(*a*) correspond to the center position of all instances of the star unit (Fig. 2*c*). The connecting formations between the main star units are determined by arrangements of overlapping star units (Fig. 2*b*). These specific arrangements, which are indicated by the gray decagons with the blue centers in Fig. 2(*d*), create patterns of two basic polygons: a hexagon and a bowtie (shown as shaded in Fig. 2*b*). The positions of the overlapping decagons are determined by the framework of the nested decagram, where the blue centers are located according to certain intersection points (Fig. 2*a*). The same overlapping arrangement around the central seed unit is also used to determine the connecting formations around the ten peripheral star units. The final line configuration of the cartwheel is shown in Fig. 2(*e*). Historically, these patterns were never merely rendered as lines. Often, the lines are thickened when incorporated into different material and sometimes broken up to suggest an interlacing pattern (Gonzalez, 2001). Fig. 2(*f*) shows the final rendered pattern of the first-level hierarchy of the global quasi-periodic empire.

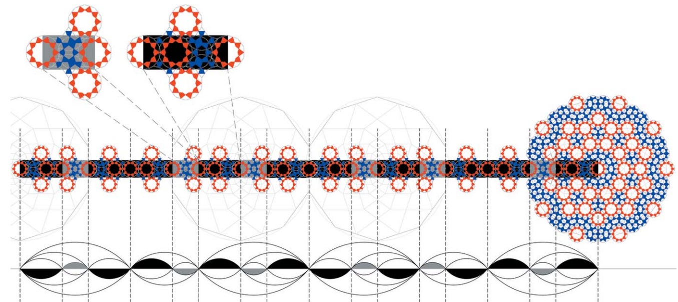


Figure 4
A closer look at one axis of the final pattern of the second-level hierarchy of the global quasi-periodic empire reveals two main clusters repeating based on the Fibonacci sequence.

It is also important to note that the connecting formations can take different internal designs without affecting the overall symmetry of the pattern. Fig. 2(*g*) shows a different possible connecting arrangement, which is constructed by extending the lines of the main star units. Figs. 2(*h*) and 2(*i*) demonstrate the process of constructing the new variation of the cartwheel pattern.

4. Growing the quasi-periodic empire

The construction of the global empire of the quasi-periodic cartwheel pattern requires building a progression of multi-level hierarchical formations. In this infinite multi-generation order, the geometric arrangement of the next higher-level order is governed by a new generation of the nested decagrams, which is derived from the same proportional system. In this sequence, the construction process of the second-level order is similar to the process of constructing the first-level order, explained earlier. The only difference is that the 'seed' unit in the second-level order is actually the final constructed cartwheel pattern of the first-level hierarchy (Fig. 2*f*). These cartwheels are distributed according to a new generation of nested decagrams (Fig. 3*a*). The black dots in Fig. 3(*a*) correspond to the center position of all instances of the cartwheel unit (Fig. 3*b*). The sequences of constructing the two main connecting formations, which are used to fill in the gaps between the main cartwheel units, are shown in Figs. 3(*c*) and 3(*d*). Following the same basic polygonal arrangements used in the first hierarchy, the internal arrangements of these

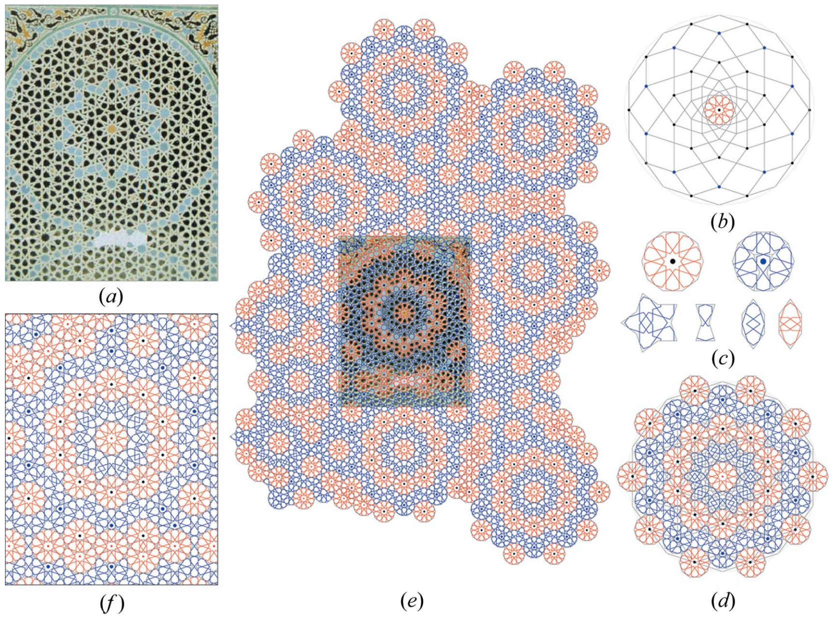


Figure 5
The sequence of constructing the cartwheel pattern of the first-level hierarchy of the quasi-periodic pattern on the interior walls of the courtyard of the Madrasa of al-'Attarin (1323), Fez, Morocco. (*a*) A photograph of the quasi-periodic pattern on the interior walls of the courtyard of the Madrasa of al-'Attarin (1323), Fez, Morocco. [Basel Kotob (1990). Courtesy of the Aga Khan Visual Archive, MIT. This material may be protected by copyright law (Title 17 US Code).] (*b*) A framework of the nested decagrams which serves as the underlying basic grid for the first-level quasi-periodic pattern. The central star 'seed' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams. (*c*) The two main star units and their different connecting formations. (*d*) The final cartwheel pattern of the first-level hierarchy of the global quasi-periodic empire. The distribution of the main units and their connecting formations are determined entirely by the network of the nested decagrams. The black and blue dots correspond to the center position of all instances of the star units. (*e*), (*f*) The overall quasi-periodic pattern on the interior walls of the courtyard of the Madrasa of al-'Attarin (1323), Fez, Morocco, is part of the second-level global empire (Fig. 6).

connecting formations are determined by combining two basic polygons: a hexagon and a bowtie (Figs. 3*c* and 3*d*). In this arrangement, the positions of all units are guided by the line decoration of the two basic polygons (Figs. 3*c* and 3*d*). Although this pattern shows one specific arrangement, other internal formations are also possible. Further research into the different possible formations and their rules is still needed. Fig. 3(*e*) shows the final pattern of the second-level hierarchy of the global quasi-periodic empire. A closer look at one axis of this empire reveals two main clusters repeating based on the Fibonacci sequence (Fig. 4).

Building on the same sequence, generating the next higher-level cluster also follows the same process, in which the new higher-generation order is built on the previous order. The final generated pattern of the previous order (Fig. 3*e*) acts as the 'seed' unit for the third-level generation order. This process can grow indefinitely to build an infinite structure of quasi-periodic formations.

5. The universal applicability of the HFM

To demonstrate the universal applicability of the HFM method to construct a variety of quasi-periodic patterns in historical Islamic architecture, consider the following two examples. The first example is the quasi-periodic pattern on the interior walls of the courtyard of the Madrasa of al-'Attarin (1323), Fez, Morocco (Fig. 5*a*). Fig. 5 demonstrates the sequence of constructing the first-level cartwheel pattern of the global empire. In this sequence, a framework of nested decagrams (Fig. 5*b*) serves as the underlying basic grid. The size of the central 'seed' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams. The black dots in Fig. 5(*b*) correspond to the center position of all instances of the red star unit and the blue dots correspond to the center position of all instances of the blue unit (Fig. 5*d*). Two different main units and four connecting formations are used (Fig. 5*c*). The red star unit has predominance over the blue main unit and the main two seed units have predominance over the connecting formations. The internal line formations of these units are flexible and can take a different design without affecting the overall symmetry. As shown in Fig. 5(*e*), the pattern on the interior walls of the Madrasa of al-'Attarin in Fez, which is spread over two cartwheels, is part of a larger quasi-periodic empire. The sequence of constructing the second-level order of the quasi-periodic empire is shown in Fig. 6. According to this sequence, the generated cartwheel in Fig.

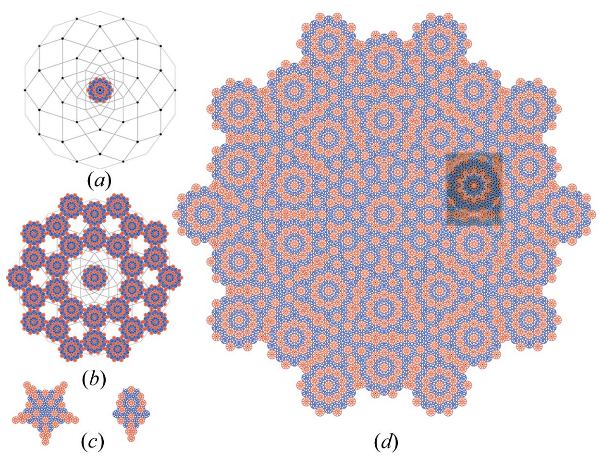


Figure 6
The sequence of constructing the second-level hierarchy of the quasi-periodic pattern on the interior walls of the courtyard of the Madrasa of al-'Attarin (1323), Fez, Morocco. (*a*) A new generation of the framework of the nested decagrams serves as the underlying basic grid for the second-level quasi-periodic pattern. The central 'seed' unit is the same final cartwheel pattern generated in Fig. 5(*d*). The black dots correspond to the center position of all instances of the main cartwheel units. (*b*) The distribution of the main cartwheel units according to the network of the nested decagrams. (*c*) The main cartwheel unit and the two connecting formations. (*d*) The overall quasi-periodic pattern on the interior walls of the courtyard of the Madrasa of al-'Attarin (1323), Fez, Morocco, is part of the second-level global empire.

5(*d*) acts as the 'seed' unit for the second-level hierarchy (Fig. 6*a*). Similar to the first-level hierarchy, the connecting formations are flexible and can take different designs without affecting the overall symmetry. However, more research into understanding the different possible formations is still needed. The final empire confirms that the pattern on the walls of the Madrasa of al-'Attarin is derived from the second-level hierarchy of the quasi-periodic empire. Building on the same process, this pattern can be expanded indefinitely.

The second case is the pattern on the external walls of the Gunbad-I Kabud tomb tower in Maragha, Iran (1197) (Fig. 7*a*). The pattern, spread over each of two adjusting panels on the walls of the Gunbad-I Kabud tomb tower, is part of a quasi-periodic cartwheel pattern (Fig. 7*a*). Fig. 7 demonstrates the sequence of constructing the first-level cartwheel pattern of the global empire. A framework of nested decagrams (Fig. 7*b*) serves as the underlying basic grid. The size of the 'seed' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams (Fig. 7*b*). Two different main units are used (Fig. 7*c*). The two units can be interchanged without affecting the overall symmetry of the pattern. The positions of all main units are determined entirely by the network of the nested decagrams (Fig. 7*d*). The connecting formations are flexible and can take different formations without affecting the overall symmetry (Fig. 7*e*). Fig. 7(*f*) shows the final cartwheel pattern of the first-level hierarchy of the global quasi-periodic empire. While Lu & Steinhardt (2007*a*,*c*) have argued that this pattern is explicitly periodic, Makovicky (2007), on the other hand, questioned if it can be expanded indefinitely. The sequence of constructing the second-level order of the quasi-periodic empire is shown in Fig. 8. According to this sequence, the generated cartwheel in Fig. 7(*f*) acts as the 'seed' unit for the second-level hierarchy (Figs. 8*a* and 8*b*). The internal arrangements of the two main connecting formations are determined by combining two basic polygonal units: a hexagon and a bowtie (Fig. 8*c*). Similar to the first-level hierarchy, these line formations are flexible and can take different designs without affecting the overall symmetry. However, as stated earlier, more research into understanding the different possible formations and their rules is needed. The construction sequence of the second-level hierarchy (Fig. 8) confirms that the pattern, spread over each of two adjusting

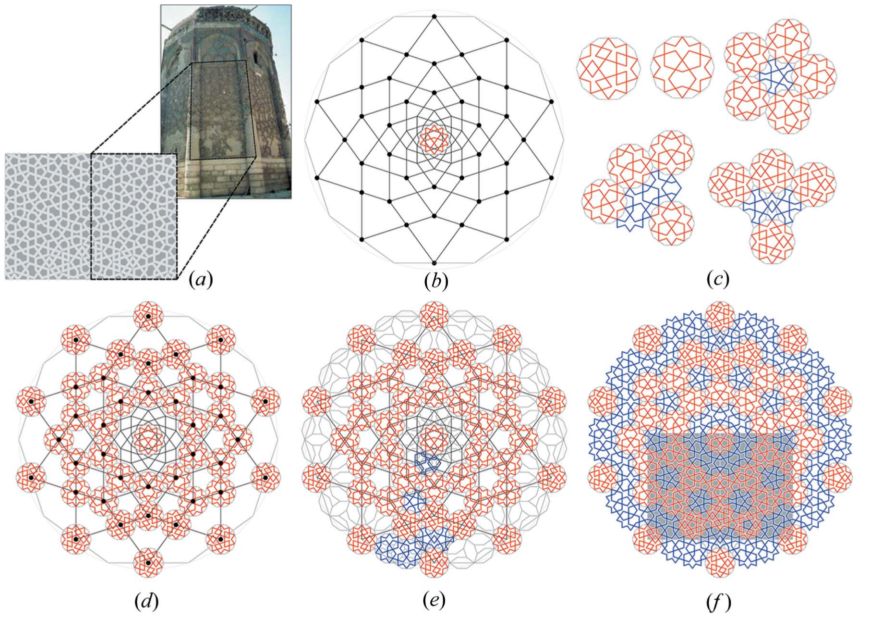


Figure 7
The sequence of constructing the cartwheel pattern of the first-level hierarchy of the quasi-periodic pattern on the external walls of the Gunbad-I Kabud tomb tower in Maragha, Iran (1197). (*a*) A photograph of the external walls of the Gunbad-I Kabud tomb tower in Maragha, Iran (1197). [Sheila Blair & Jonathan Bloom (1984). Courtesy of the Aga Khan Visual Archive, MIT. This material may be protected by copyright law (Title 17 US Code).] The pattern, spread over each of two adjusting panels on the walls of the Gunbad-I Kabud tomb tower in Maragha, is part of the first-level cartwheel pattern. (*b*) A framework of the nested decagrams serves as the underlying basic grid for the first-level quasi-periodic pattern. The central star 'seed' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams. (*c*) The two main star units and their different connecting formations. (*d*), (*e*) The distribution of the main units and their connecting formations are determined entirely by the network of the nested decagrams. The black dots correspond to the center position of all instances of the two main units. These can be interchanged without affecting the overall symmetry. (*f*) The final cartwheel pattern of the first-level hierarchy of the global quasi-periodic empire.

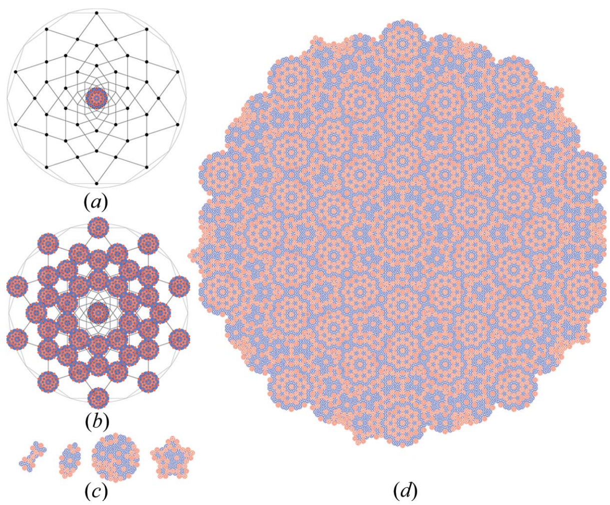


Figure 8
The sequence of constructing the second-level hierarchy of the quasi-periodic pattern on the external walls of the Gunbad-I Kabud tomb tower in Maragha, Iran (1197). (*a*) A new generation of the framework of the nested decagrams serves as the underlying basic grid for the second-level quasi-periodic pattern. The central 'seed' unit is the same final cartwheel pattern generated in Fig. 7(*f*). (*b*) The distribution of the main cartwheel units according to the network of the nested decagrams. (*c*) The different connecting formations. (*d*) The final pattern of the second-level hierarchy of the global quasi-periodic empire.

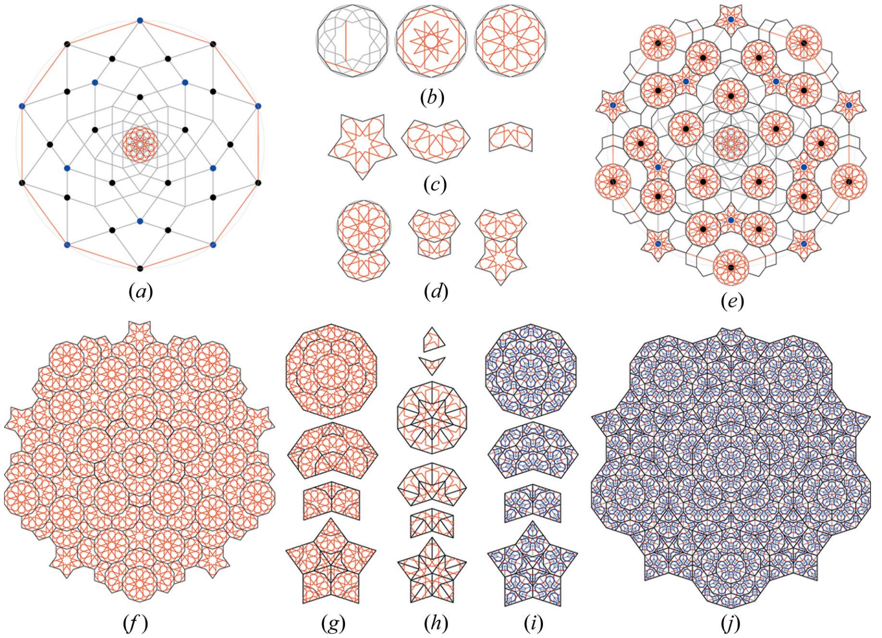


Figure 9
The sequence of constructing the first-level hierarchy of the new quasicrystalline pattern. (*a*) A framework of the nested decagrams, which serves as the underlying basic grid for the first-level quasicrystalline empire. The central star 'seed' unit is proportional to the size of the framework and is strictly derived from the diminution sequence of the nested decagrams. (*b*) The main 'seed' unit is created by proportionally breaking down the decagon to form the final designs. (*c*) The three different connecting formations are fragments of the main seed unit. (*d*) The different combinations between the main units and the connecting formations. (*e*) The distribution of the main units and their connecting formations are determined entirely by the network of the nested decagrams. The black dots correspond to the center position of all instances of the main units. (*f*) The final cartwheel pattern of the first-level hierarchy of the global quasicrystalline empire. (*g*) The inflation rules for the new generated pattern. (*h*) The new pattern can be mapped perfectly to Penrose tiling patterns; the kite and dart with their new decoration. (*i*) The result from mapping the same inflation rules of the new pattern to Penrose tiles. (*j*) Perfect Penrose formations created by applying the inflation rules in Fig. 9(*i*).

panels on the walls of the Gunbad-I Kabud tomb tower in Maragha, is in fact part of a global quasi-periodic empire and can be expanded indefinitely.

6. Constructing new quasicrystalline formations

One important characteristic of using the hierarchical proportional framework is that it allows for the construction of a wide variety of fivefold and tenfold quasicrystalline patterns. This is achieved by changing two parameters: the internal design of the 'seed' units and the distribution of these units by the framework. To illustrate this point, consider the example shown in Fig. 9. Fig. 9(*a*) shows the same framework used in the previous example, but with a different distribution of 'seed' units. The seed units are created by proportionally breaking down the decagon to form the final designs (Fig. 9*b*). The new distribution of these units is shown in Fig. 9(*e*). The connecting formations between the main units are actually fragments of the main seed unit and are shown in Fig. 9(*c*) and their combinations are shown in Fig. 9(*d*). The final reconstructed pattern of the first-level hierarchy is shown in Fig. 9(*f*).

Derived from this process, it is possible to deduce the inflation rules for this pattern (Fig. 9*g*). The new pattern can be mapped perfectly to Penrose tiling systems. Fig. 9(*h*) shows the kite and dart with the new decoration. Fig. 9(*i*) shows the result from mapping the same inflation rules of the new pattern to Penrose tiles. The result from applying the same inflation rules creates perfect Penrose formations (Fig. 9*j*).

Constructing the global empire of the new pattern follows the same process described in the previous examples (Fig. 10). The 'seed' unit of the second-level hierarchy is the same final generated pattern of the first-level order, shown in Fig. 9(*f*). The connecting elements (Fig. 10*c*) are fragments of the new 'seed' unit. The final generated pattern of the second-level hierarchy of the global quasicrystalline empire is shown in Fig. 10(*d*).

An important characteristic of using this multi-level proportional system is that the same elements of the patterns recur at different scales. Mathematicians often describe this as the 'self-similarity' principle, which is the key to understanding the geometrical similarities between these formations and structures in the natural world, including quasicrystals. A close-up detail of the generated empire in Fig. 10(*d*) reveals the second-level inflation rules. The same rules can be deduced from the global empire of perfect Penrose tiling (Fig. 11). A detailed description of how to use the HFM to

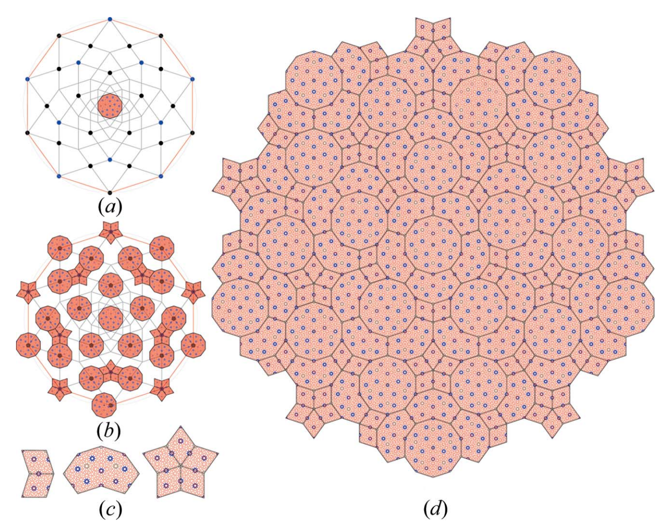


Figure 10
The sequence of constructing the second-level hierarchy of the new quasicrystalline pattern generated in Fig. 5. (*a*) A new generation of the framework of the nested decagrams serves as the underlying basic grid for the second-level quasicrystalline pattern. The central 'seed' unit is the same final cartwheel pattern generated in Fig. 9(*f*). (*b*) The distribution of the main cartwheel units according to the network of the nested decagrams. (*c*) The different connecting formations. (*d*) The final pattern of the second-level hierarchy of the global quasicrystalline empire.

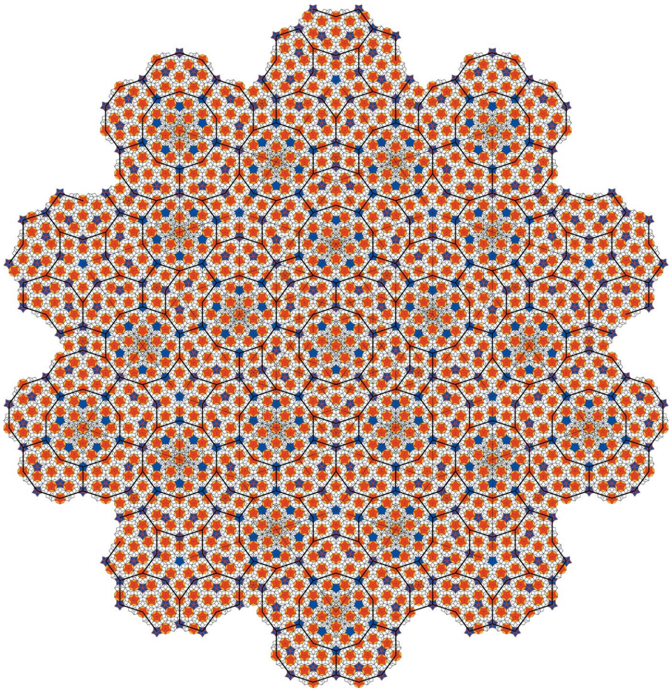


Figure 11
The second-level hierarchy of the global quasicrystalline empire of a Penrose tiling pattern. The pattern shows that the second-level inflation rules are the same as the inflation rules shown in Fig. 10(*d*).

construct the global empires of two perfect Penrose tiling systems can be found in Al Ajlouni (2011).

The most striking quality arising from the application of this specific formation is that it allows the distribution of an indefinitely large number of elements with a regular spacing, a property described as long-range order. This quality is evident through the distribution of the main units within the global empire (Figs. 10*d* and 11). The distances between these units are multiplications of φ (1.618) and 1 + φ (2.6180).

7. Conclusions

In this paper, I have presented a global construction model (HFM) that is able to describe the long-range translational and orientational order of quasi-periodic formations in Islamic architecture. This model works in perfect concert with the conceptual framework, philosophy and '*conventional*' methods, which, for centuries, have generated this carefully calculated art form. The evidence presented in this paper suggests that the quasi-periodic patterns in Islamic architecture were conceived and constructed through a global system and not based on local rules (*e.g.* tiling, subdividing or overlapping). In this way, quasi-periodic formations can grow rapidly *ad infinitum* without the need for any defects or mismatches. Moreover, the findings of this paper suggest that ancient Muslim designers, by using the most primitive tools (a compass and a straightedge), were able to unlock the mysteries of the global long-range order of quasicrystalline formations, which the West is still struggling to resolve.

This new method, which can be used as a general guiding principle for constructing new quasi-periodic formations, could possibly provide a deeper understanding of the structure of quasicrystals at an atomic scale, allowing scientists to achieve improved control over their composition and structure, potentially leading to the development of new materials and devices. In addition, this novel method provides an easy tool for mathematicians, teachers, designers and artists to generate and study these complicated quasi-periodic symmetries.

Future efforts should be directed toward investigating the application of similar hierarchical global methods to generate other perfect quasicrystalline formations of seven-, eight-, nine-, 11- and 12-fold symmetries, and how these abstract geometric models can actually correlate with real quasi-periodic structures.

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